PHYSICAL REVIEW VOLUME 131, NUMBER 6 15 SEPTEMBER 1963

Generalization of Mass Formula in Unitary Symmetries*

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Okubo's mass formula for unitary symmetries is generalized to any order. It is shown that the number of terms in a formula for a representation true to all orders is exactly the same as the number of isotopic multiplets contained in the representation. This result also holds for strong interaction symmetries based on other rank-2 groups.

I. INTRODUCTION

ONE of the convincing features of unitary symmetry models[1,2] for strong interactions of elementary particles has been the success of the first-order mass relation derived by Okubo.[3] His recent derivation of the second-order relation[4] has prompted us to obtain its generalization to any order. In Sec. II, we define the medium-strong interaction to order n [Eq. (2.2)] which allows us to write the generalization almost immediately (Sec. III). For particular representations, however, simplification is achieved through the property of the interaction that all the components of the irreducible tensors appearing in its reduction commute with the operators N, S, and $\mathbf{I}$. From this we conclude (Theorem I of Sec. IV) that these tensors must be of the symmetry type $(f,\ 0,\ -f)$. Corresponding to the multiplicity $d_f(\leqslant f+1)$ of this representation in the reduction of the direct product of a representation D and its contragradient $\bar{D}$, there exist d_f linearly independent tensor operators of the above symmetry type that can be constructed from the generators and give rise to the only nonvanishing matrix elements contributing to the mass formula. In Appendix II we are able to select these from a set of $f+1$ that are added to the mass formula at the fth stage. This gives us the formula for a particular representation D to any given order n (Sec. IV). A simple corollary is the *exact* relation for it which holds *to all orders*. This contains the same number of terms as the number of isotopic multiplets in the basis for the representation. Section V deals with the consequences.

II. SYMMETRY BREAKING INTERACTION

In the unitary symmetry models of Sakata[2] and Gell-Mann–Ne'eman,[1] the strong interaction Lagrangian is considered to be invariant under the groups $U(3)$ and $SU(3)/C_3$, respectively. This results in the classification of elementary particles as degenerate supermultiplets which form bases for the irreducible representations of these groups. If the symmetries were exact, all the members of these supermultiplets would have the same mass. However, this does not appear to be the case in nature and attempts have been made to break these symmetries in a manner which removes the mass degeneracy and gives the correct mass spectrum. These attempts are primarily based upon Pais'[5] philosophy of a hierarchy of interactions in which a sequence of very strong, medium strong, electromagnetic, etc. interactions are assumed to exist in nature with progressively weaker symmetries. In other words, the interaction Lagrangian can be written as

$$I_{\mathrm{vs}}+I_{\mathrm{ms}}+I_{\mathrm{em}}+\cdots,$$

where I_{vs}, the very strong part, is invariant under the full symmetry group [$U(3)$ or $SU(3)/C_3$ in the unitary symmetry models]; the medium strong I_{ms} under a subgroup of the full group which in turn includes the subgroup that leaves the electromagnetic interaction I_{em} invariant. Since I_{ms} and I_{em} respect only a part of the full symmetry, their application will remove the mass degeneracy in two stages: In the first stage when the medium-strong interaction is switched on, the super multiplets subdivide into isotopic multiplets which contain a number of degenerate entities; turning on the electromagnetic interaction completely removes the degeneracy, resulting in mass splittings between all the members of the supermultiplets.

We concern ourselves with the first stage only. If no restriction is imposed on the form of the medium-strong interaction (sometimes called the symmetry-breaking interaction in the text) obviously no progress whatsoever can be made. As our goal at this stage is to break the supermultiplets into isotopic multiplets, we suppose that the medium-strong interaction is an operator T that commutes with the isotopic spin, strangeness, and nucleon-number operators $\mathbf{I}$, S, N, respectively, (Assumption I). This restriction is highly reasonable as we are still in the realm of strong interactions where strangeness and nucleon number are conserved, and any noncommutation with $\mathbf{I}$ will result in mass splittings between different members of the isotopic multiplets.

The above restriction alone is still not sufficient for our purpose. We therefore make the further assump-

* The research reported in this document has been sponsored in part by Air Force Office of Scientific Research, OAR, through the European Office, Aerospace Research, U. S. Air Force.

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[2] M. Ikeda, S. Ogawa, and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) **22**, 715 (1959); Y. Yamaguchi, Suppl. Progr. Theoret. Phys. (Kyoto) **11**, 1 (1959); J. Wess, Nuovo Cimento, **10**, 15 (1960).

[3] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[4] S. Okubo, Phys. Letters **4**, 14 (1963).

[5] A. Pais, Phys. Rev. **86**, 633 (1952).

tion[6] (in analogy with electromagnetism) that the operator, to lowest order, transforms, as the adjoint representation of the group[7] (Assumption II). These conditions then fix the operator (to lowest order) as the $T_3{}^3$ component of a tensor $T_\nu{}^\mu$. This tensor $T_\nu{}^\mu$, in $SU3/C_3$ is irreducible. However, in $U(3)$, as the adjoint representation is reducible, we can write it as

$$T_\nu{}^\mu=\delta_\nu{}^\mu+M_\nu{}^\mu\,, \tag{2.1}$$

where $M_\nu{}^\mu$ is irreducible and transforms as the 8-dimensional representation.

Henceforth, we shall confine ourselves to the $U(3)$ scheme which is simpler to deal with. The results, however, apply to both the schemes.

To nth order, we take this operator to be

$$T_n=T_3{}^3+T_3{}^3T_3{}^3+\cdots+T_3{}^3T_3{}^3\cdots T_3{}^3\;{}^{(n\text{ factor})}$$
$$=\sum_{i=1}^{n}\prod_i T_3{}^3\,, \tag{2.2}$$

where

$$\prod_i T_3{}^3=T_3{}^3T_3{}^3\cdots T_3{}^3\;{}^{(i\text{ factors})}. \tag{2.3}$$

Since every product of tensors $T_\nu{}^\mu$ is reducible under $U(3)$, we can express (2.3) as

$$\prod_i T_3{}^3=\sum_{r=0}^{i}a_r(\delta_3{}^3)^r M_{33\cdots 3}{}^{33\cdots 3\,(i-r\text{ times})}\,,$$

where $M_{33\cdots 3}{}^{33\cdots 3\,(k\text{ times})}$ is a component of an irreducible tensor.

Thus,

$$T_n=\sum_{i=1}^{n}\sum_{\gamma=0}^{i}a_\gamma(\delta_3{}^3)^r M_{33\cdots 3}{}^{33\cdots 3\,(i-r\text{ times})}. \tag{2.4}$$

From our first assumption

$$[T_3{}^3,\mathbf{I}]=[T_3{}^3,S]=[T_3{}^3,N]=0$$

and Eq. (2.1) above, it follows that $M_3{}^3$ also commutes with $\mathbf{I}$, N, S. By induction now

$$[M_{3\cdots 3}{}^{3\cdots 3},\mathbf{I}]=[M_{3\cdots 3}{}^{3\cdots 3},S]=[M_{3\cdots 3}{}^{3\cdots 3},N]=0. \tag{2.5}$$

We note that Okubo's expression $T_3{}^3+T_{33}{}^{33}$ is equivalent to our $T_3{}^3+T_3{}^3T_3{}^3$ as both of these reduce to

$$a\delta_3{}^3+bM_3{}^3+cM_{33}{}^{33}.$$

III. GENERALIZATION OF OKUBO'S MASS FORMULA

$U(3)$ has nine generators $A_\nu{}^\mu$ which satisfy the commutation relations

$$[A_\nu{}^\mu,A_\beta{}^\alpha]=\delta_\nu{}^\alpha A_\beta{}^\mu-\delta_\beta{}^\mu A_\nu{}^\alpha. \tag{3.1}$$

The irreducible representations $D\equiv(f_1,f_2,f_3)$ are characterized by 3 integers f_1, f_2, f_3 such that

$$f_1\geqslant f_2\geqslant f_3. \tag{3.2}$$

$\bar{D}\equiv(-f_3,-f_2,-f_1)$ is the representation contragradient to D.

From now onwards the word "representation" will refer only to an irreducible representation.

To obtain the different values of I and S contained in the basis of a representation $D\equiv(f_1,f_2,f_3)$, we determine all pairs of numbers f_1', f_2' such that[8]

$$f_1\geqslant f_1'\geqslant f_2\geqslant f_2'\geqslant f_3. \tag{3.3}$$

Then we have

$$I=\tfrac{1}{2}(f_1'-f_2') \tag{3.4}$$

and

$$S=f_1'+f_2'-(f_1+f_2+f_3). \tag{3.5}$$

Note. N in the Sakata model [$U(3)$ scheme] is given by

$$N=f_1+f_2+f_3. \tag{3.6}$$

However, it is outside the symmetry group $SU(3)/C_3$ in the Gell-Mann–Ne'eman model.

A tensor $T_\nu{}^\mu$ transforming as the adjoint representation satisfies the commutation relation

$$[A_\beta{}^\alpha,T_\nu{}^\mu]=\delta_\beta{}^\mu T_\nu{}^\alpha-\delta_\nu{}^\alpha T_\beta{}^\mu. \tag{3.7}$$

Lemma I. In any representation

$$\underbrace{(AAA\cdots A)}_{n\text{ factors}}{}_{33\cdots 3}{}^{33\cdots 3\,(m\text{ times})}$$
$$=\sum_{\substack{r,s,t\geqslant 0\\ r+s+t=m}}a_{rst}(\delta_3{}^3)^r(A_3{}^3)^s((AA)_3{}^3)^t. \tag{3.8}$$

Proof. We have

$$[A_3{}^3,A_3{}^3]=[A_3{}^3,(AA)_3{}^3]=[(AA)_3{}^3,(AA)_3{}^3]=0.$$

Since the Casimir operators $\langle AA\rangle$, $\langle AAA\rangle$ commute with the generators $A_\nu{}^\mu$, the lemma follows from Eq. (A10) in Ref. 3 on replacing $T_\nu{}^\mu$ by $A_\nu{}^\mu$.

Theorem I. The mass formula to order n for *every* representation is a sum of $\frac{1}{2}[(n+1)(n+2)]$ terms, and can be written as

$$M^n=\sum_{i=0}^{n}\sum_{j=0}^{i}a_{ij}[I(I+1)-\tfrac{1}{4}S^2]^jS^{i-j}\,, \tag{3.9}$$

where a_{ij} are parameters depending upon the representation but independent of the subquantum numbers I and S.

Proof. From Eq. (2.2) we have

$$M^n=\langle D,\psi|T_n|D,\psi\rangle,$$

where D is any arbitrary representation of $U(3)$ and ψ any vector in its basis.

As the associative algebra $\mathcal{A}$ generated by the

[6] This assumption is the same as that made by Gell-Mann (Ref. 1) and Okubo (Ref. 3).

[7] Transformation law for a tensor belonging to the adjoint representation is given in Eq. (3.7).

[8] H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1939).

infinitesimal generators of $U(3)$ within a representation is the whole matrix algebra over the representation space,[9] we can write the above matrix element for M^n as a sum of matrix elements of suitable operators constructed from the generators of the group. Thus, by Lemma I

$$M^n=\sum_{i=0}^{n}\sum_{j=0}^{i}\langle D,\psi|b_{ij}(A_3{}^3)^{i-j}[(AA)_3{}^3]^j|D,\psi\rangle.$$

The mass formula follows from this when we write

$$A_3{}^3=-S$$
$$(AA)_3{}^3=[I(I+1)-\tfrac{1}{4}S^2]+aS+b,$$

where a and b are independent of I and S.

IV. SPECIALIZATION TO PARTICULAR REPRESENTATIONS

In the case of the 10-dimensional representation Gell-Mann[10] remarked that the first-order formula

$$M_{10}{}^1=a+bS+c[I(I+1)-\tfrac{1}{4}S^2]$$

reduces to

$$M_{10}{}^1=a'+b'S$$

on account of the relation

$$I=1+\tfrac{1}{2}S.$$

We wish to point out that the second-order formula[4]

$$M^2=a+bS+c[I(I+1)-\tfrac{1}{4}S^2]+dS^2+eS[I(I+1)-\tfrac{1}{4}S^2]+f[I(I+1)-\tfrac{1}{4}S^2]^2,$$

when applied to the 8-dimensional representation becomes

$$M_8{}^2=a'+b'S+c'[I(I+1)-\tfrac{1}{4}S^2]+d'S^2$$

as a result of the relations

$$SI(I+1)=\tfrac{3}{4}S$$
$$S^3=S$$
$$I(I+1)[I(I+1)-2]=-\tfrac{15}{16}S^2.$$

In order to see when and why this happens, we shall look at the formula from a different point of view. In Eq. (2.4) we expressed T_n, the symmetry-breaking interaction to nth order, as a sum of components of irreducible tensors. Each one of these components commutes with the operators N, S and $\mathbf{I}$ [see Eq. (2.5)]. Therefore, these appear only in the irreducible tensors which correspond to the representations (with $N=0$) containing in their bases an isotopic multiplet with $I=S=0$. The representations can only be of the form $(f,0,-f)$ as we prove below.

Lemma II. In representations (f_1, f_2, f_3) with $N=0$, the isotopic multiplet $I=S=0$ occurs only when $f_3=-f_1$, $f_2=0$.

Proof. From Eqs. (3.4), (3.5), and (3.6) we obtain on setting $I=S=N=0$,

$$f_1'=f_2',\quad f_1'=-f_2',$$

i.e., $f_1'=f_2'=0$. Now using (3), $f_2=0$. Finally from (3.6), $f_3=-f_1$.

Remark. The group $SU3/C_3$ in the Gell-Mann–Ne'eman model has the representations (f_1, f_2, f_3) with a restriction which may be taken as

$$f_1+f_2+f_3=0.$$

So the lemma holds equally well though N is outside the symmetry group.

We have seen that the irreducible tensors in the expression of the symmetry-breaking operator all belong to the representations of the form $(f,0,-f)$. Now we consider the reduction of the direct product $D\otimes\bar{D}$,[11] in which we see that the representation $(f,0,-f)$ occurs at most $f+1$ times. (This is a special case of Theorem A.I proved in Appendix A.) This gives us the $f+1$ terms in the formula (3.9) which were added at the fth stage. However for a particular D, the representation $(f,0,-f)$ may not occur $f+1$ times in the reduction of $D\otimes\bar{D}$. (See Theorem A.I of Appendix A.) When this happens there exist relations which allow a reduction in the number of terms added at the fth stage to *exactly* the number of times the representation $(f,0,-f)$ occurs in the reduction $D\otimes\bar{D}$. (See lemmas in Appendix B and Theorem A.I.) Hence, the mass formula to order n applicable to a particular representation D will be

$$M_D{}^n=\sum_{f=0}^{n}\sum_{j=0}^{d_f-1}a_{fj}[I(I+1)-\tfrac{1}{4}S^2]^jS^{f-j},\qquad(4.1)$$

when d_f is the number of times the representation $(f,0,-f)$ occurs in the reduction $D\otimes\bar{D}$.

It is also clear from Theorem A.I that the representations $(f,0,-f)$ with $f>f_1-f_3$ do not occur at all in the reduction of the direct product $D\otimes\bar{D}$. From this it follows that an *exact* formula for D (true to *all* orders) is

$$M_D=M_D{}^{\mu+\nu}\quad(\mu=f_1-f_2,\ \nu=f_2-f_3).\qquad(4.2)$$

The total number of terms in this formula is equal to

$$\sum_{f=0}^{\mu+\nu}d_f\qquad(4.3)$$

which by Theorem A.I is also

$$(\mu+1)(\nu+1).\qquad(4.4)$$

Now as each one of these representations $(f,0,-f)$ contains $I=S=0$ just once, $(\mu+1)(\nu+1)$ is also equal to

[9] This follows from Schur's lemma. See also J. Ginibre in Ref. 11.

[10] M. Gell-Mann, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN Scientific Information Service, Geneva, 1962), p. 805. See also, S. L. Glashow and J. J. Sakurai, Nuovo Cimento **26**, 622 (1962).

[11] This approach is similar to that employed in the following: N. Cabibbo and R. Gatto, Nuovo Cimento **21**, 872 (1962); B. Diu (to be published); J. Ginibre (to be published).

the number of times $I=S=0$ multiplets occur in the direct product of D and $\bar{D}$. We now prove

Theorem II. The number of times $I=S=0$ multiplets occurs in the direct product $D\otimes\bar{D}$ is equal to the number of isotopic multiplets in the representation D.

Proof. The representations D and $\bar{D}$ consist of isotopic multiplets of the form (I,S), (I',S') where the set of $I'(S')$ is the same as that of $I(-S)$. From (I,S) and (I',S') we obtain

$$\sum_{I=|I-I'|}^{I+I'} (I,\, S+S').$$

This series contains $I=S=0$ if and only if

$$S+S'=0=I-I',$$

i.e., when $S'=-S$, $I'=I$. Therefore, there exists a unique solution for (I',S') for each (I,S) which satisfies the theorem.

Thus, each of the multiplets (I,S) of D gives rise to one and only one method of constructing $I=S=0$ in the direct product $D\otimes\bar{D}$.

Finally, we conclude from the theorem that the mass formula M_D has exactly the same number of parameters as the number of isotopic multiplets in D.

It is clear that the above result also holds for strong interaction symmetries based on other rank-2 groups.

V. CONCLUSION

The conclusion reached at the end of the last section about the number of parameters is not unexpected as the group-theoretic approach gives the kinematical structure only. The detailed dynamics is contained in the values of these parameters and future efforts should be directed towards an understanding of their relationship with the spatial properties like spin and parity, etc.

An interesting feature is the observation that the mass formula to first order when applied to baryon and pseudoscalar meson octets gives a good fit. The same probably happens in the case of the baryon-meson resonances forming the 10-dimensional representation. A calculation based on 1238-, 1385-, and 1535-MeV masses of N^*, Σ_1, Ξ_1 predicts the mass of the fourth $p^{3/2}$ resonance as 1680 MeV. Should such a resonance be found, the values of the parameters c and d in the general formula

$$M_{10}=a+bY+cY^2+dY^3 \quad (Y=N+S)$$

would be almost zero. Although the calculations above are not perturbation theoretic, the rapid convergence of the "series" presents us with a puzzle. It is a useful conjecture to consider some of the last few parameters in an exact formula to be zero as a starting point for predicting the position of new resonances and their assignment to various representations. The case of the vector-meson octet appears to be somewhat involved.

ACKNOWLEDGMENTS

The authors are indebted to Professor Abdus Salam for suggesting this work and for constant encouragement. We would also like to thank Dr. A. R. Edmonds for reading the manuscript and for many helpful comments. One of us (M.A.R.) is also grateful to the British Department of Technical Cooperation and the Pakistan Atomic Energy Commission for the grant of a fellowship under the Colombo Plan.

APPENDIX A

Theorem A.I. In the reduction of the direct product of a representation $D\equiv(f_1,f_2,f_3)$ with its contragradient $\bar{D}\equiv(-f_3,\,-f_2,\,-f_1)$ the representation $(f,\,0,\,-f)$ occurs d_f times, where

$$d_f=\begin{cases} 0 & \text{when } \mu+\nu<f \\ \mu+\nu-f+1 & \text{when } \mu<f,\quad \nu<f \text{ but } \mu+\nu\geqslant f \\ \mu+1 & \text{when } \mu<f,\quad \nu\geqslant f \\ \nu+1 & \text{when } \mu\geqslant f,\quad \nu<f \\ f+1 & \text{when } \mu\geqslant f,\quad \nu\geqslant f \end{cases}$$

and $\mu=f_1-f_2$, $\nu=f_2-f_3$.

Proof. The theorem can be proved by working with the characters. However, we use the much simpler procedure of multiplying Young's tableau. As some of the integers labeling the representations D and $\bar{D}$ are negative, we first of all consider the representations

$$D_1\equiv(f_1-f_3,\,f_2-f_3,\,0)\equiv(\mu+\nu,\,\nu,\,0)\,,$$
$$\bar{D}_1\equiv(f_1-f_3,\,f_1-f_2,\,0)\equiv(\mu+\nu,\,\mu,\,0)\,.$$

The corresponding Young's tableau for D_1 ($\bar{D}_1$) has $\mu+\nu$ squares in the first row and ν (μ) squares in the second. We are interested in the representation $(f,\,0,\,-f)$ in the product $D\otimes\bar{D}$. As D_1 ($\bar{D}_1$) has been obtained from D ($\bar{D}$) by subtracting f_3 ($-f_1$) from each of the three integers labeling the representation, we should look for the representation $(f,\,0-f)$ in the product $D_1\otimes\bar{D}_1$ as associated with the Young's tableau

$$(f+f_1-f_3,\,f_1-f_3,\,-f+f_1-f_3) \\ =(\mu+\nu+f,\,\mu+\nu,\,\mu+\nu-f)\,.$$

To obtain the product diagrams (see Fig. 1), we write α's (β's) in the squares in the first (second) row of the

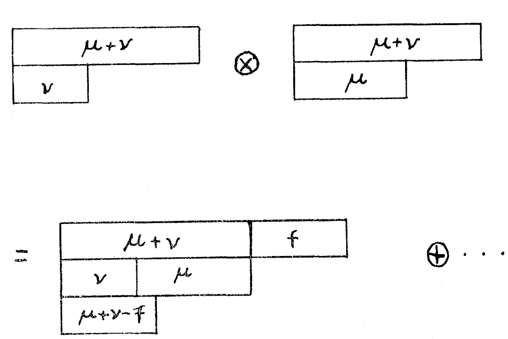

FIG. 1. Decomposition of $D_1\otimes\bar{D}_1$ by Young's Tableau.

diagram $\tilde{D}_1$ and then adjoin these squares containing α's and β's, in this order, to the diagram D_1 such that (i) the final diagram has always $\leqslant 3$ rows; (ii) when we finish adjoining squares containing α's (β's), it is a Young's tableau; (iii) the adjoined α's and β's when read from the right, exhausting the first row first, and then the second, etc., form a lattice order, i.e., at each stage in this order the number of α's is not less than the number of β's.

In this product we are interested in the diagrams with $\mu+\nu+f$, $\mu+\nu$, $\mu+\nu-f$ squares in the first, second, and third rows, respectively. This is obtained by adding f squares containing α's to the first row of D_1 followed by μ, $\mu+\nu-f$ squares containing some α's and some β's to the second and third rows in a manner that satisfies the 3 conditions stated above. The condition (ii), requires β additions in the second and third rows to be always on the right of all α's. The condition (iii) of lattice order says that the number of β's to be added to the second row must be $\leqslant f$. Thus, the number of diagrams of the above type in the product can be at most $f+1$ (corresponding to 0, 1, 2, $\cdots f$ number of β's added to the second row).

However, all these cases are not *always* possible. To examine this carefully, let us first consider μ. If $\mu\geqslant f$ all the $f+1$ cases *might* be possible. But when $\mu<f$ only $\mu+1$ of these cases (which correspond to 0, 1, 2, $\cdots\mu$ addition of β's to the second row) are possible. All these cases will definitely be possible if we can fill all the squares in the second row with the rest of the α's. As there are only $\mu+\nu-f$ squares to be adjoined to the third row, this requires $\mu+\nu-f\geqslant\mu$ or $\nu\geqslant f$. On the other hand, when $\mu+\nu-f<\mu$ or equivalently $\nu<f$, then $\mu-(\mu+\nu-f)=f-\nu$ β's (at least) will have to be added to the second row. This will reduce the number of possibilities in each of the above cases by exactly $f-\nu$ to $(f+1)-(f-\nu)=\nu+1$ and $(\mu+1)-(f-\nu)=\mu+\nu-f+1$, respectively. Since the condition (iii) is also satisfied by each one of these cases, the theorem follows.

APPENDIX B

Mass Formula for a Particular Representation

To derive the mass formula $M_D{}^n$ for a representation D we need the following lemmas:

Throughout this Appendix we take $D\equiv(f_1,f_2,f_3)$ and $\mu=f_1-f_2\geqslant\nu=f_2-f_3$.

Lemma B.I. In any irreducible representation D of $U(3)$:

(i) I takes the $\mu+\nu+1$ distinct values 0, $\frac{1}{2}$, 1, $\cdots$, $\frac{1}{2}(\mu+\nu)$ with multiplicities 1, 2, $\cdots\nu$, $\nu+1$, $\nu+1$, $\cdots\nu+1$, ν, $\nu-1$, $\cdots 1$, respectively.

(ii) S takes the $\mu+\nu+1$ distinct values f_1+f_2-n, f_1+f_2-1-n, $\cdots$, f_2+f_3-n with multiplicities 1, 2, $\cdots\nu$, $\nu+1$, $\nu+1$, $\cdots\nu+1$, ν, $\cdots$, 1, respectively, with $n=f_1+f_2+f_3$.

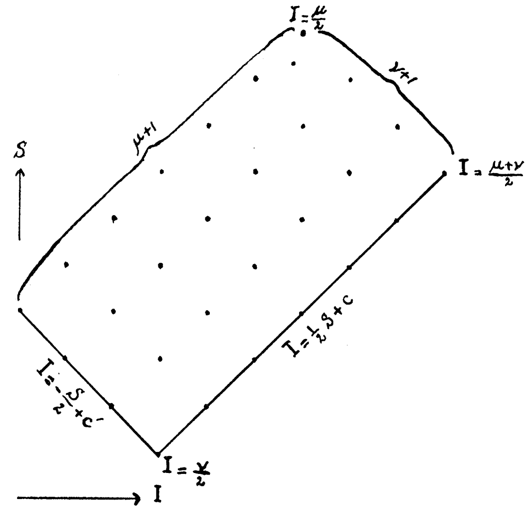

FIG. 2. The (I,S) plot for a representation (f_1,f_2,f_3). $(\mu=f_1-f_2\leqslant\nu=f_2-f_3.)$

The proof follows from Eqs. (3.3), (3.4), and (3.5).

Lemma B.II. The points (I,S) corresponding to the isotopic multiplets in D form a lattice consisting of $\nu+1$ $(\mu+1)$ equally spaced parallel lines with equations of the form

$$I=\tfrac{1}{2}S+c\,(I=-\tfrac{1}{2}S+c')\,.$$

Proof. Eliminating f_1' and f_2' in turn from Eqs. (3.4) and (3.5) we obtain

$$I=\tfrac{1}{2}S-f_2'+\tfrac{1}{2}(f_1+f_2+f_3) \tag{B1}$$

$$I=-\tfrac{1}{2}S+f_1'-\tfrac{1}{2}(f_1+f_2+f_3)\,. \tag{B2}$$

Corresponding to $\nu+1$ $(\mu+1)$ different fixed values of f_2' (f_1') (B1) and (B2) are the equations referred to in the lemma.

From Lemmas B.I and B.II we can construct the following lattice of points (I,S) for the representation D (Fig. 2).

Lemma B.III. If A and B are any functions satisfying

$$\text{(i)}\qquad A^{\nu+1}=\sum_{i=1}^{\nu+1}\alpha_iA^{\nu-i+1}B^i+\sum_{\substack{i+j\leqslant\nu+1\\ i,\,j\geqslant 0}}\alpha_{ij}A^iB^j \tag{B3}$$

and

$$\text{(ii)}\qquad A^iB^{\mu+\nu+1-2i}+\sum_{j=0}^{\nu-1}\sum_{k=0}^{j}\alpha_{ijk}A^kB^{\mu+\nu-j-k}+\sum_{j=0}^{\mu-\nu-1}\sum_{k=0}^{\nu}\beta_{ijk}A^kB^{\nu+1+j-k}+\sum_{\substack{j,\,k\geqslant 0\\ j+k<\nu+1}}\gamma_{ijk}A^jB^k \tag{B4}$$

for $i=0, 1, 2, \cdots\nu$, then all other expressions of the form $A^\alpha B^\beta(\alpha,\beta\geqslant 0)$ not included in the above equations are

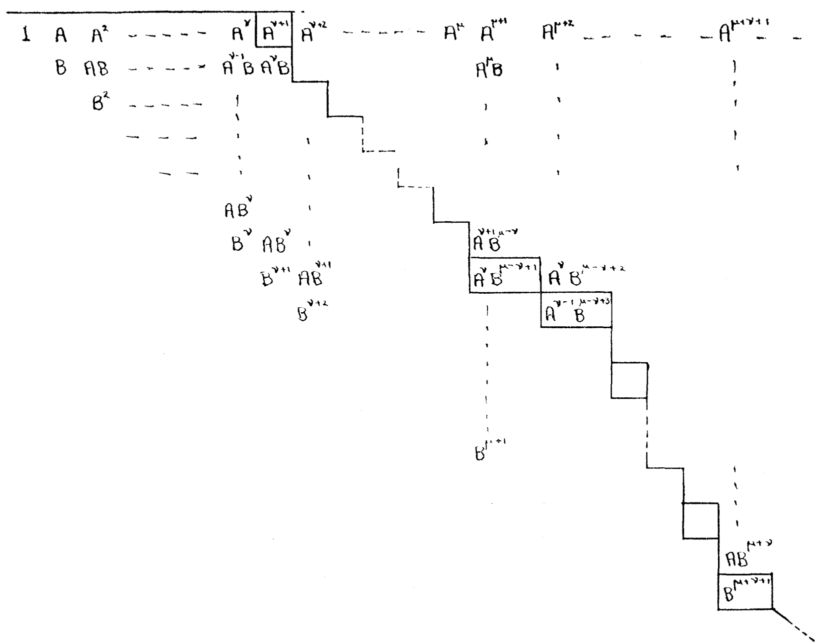

FIG. 3. Schematic representation of terms appearing in the mass formula.

also expressible in terms of quantities on the right in Eq. (B4).

The quantities $A^{\nu+1}$, $A^iB^{\mu+\nu+1-2i}$ appearing on the left-hand side of the above equations are enclosed in squares in Fig. 3. Equation (B3) has on its right-hand side quantities in the $(\nu+1)$st column except $A^{\nu+1}$ and those on the left of this column. The quantities on the right-hand side of the Eqs. (B4) are all the quantities below the zig-zag line. The content of the lemma is that all the quantities in the figure above the zig-zag line and not in the squares are expressible in terms of the ones below this line provided that the quantities in the squares satisfy (B3) and (B4).

The proof of the above assertion is trivial: We start with the expression (B3) for $A^{\nu+1}$ and multiply it first by B and then by A obtaining expressions for $A^{\nu+1}B$ and $A^{\nu+2}$ in terms of the quantities on the right-hand side of (B3) and the ones in $\nu+2nd$ column below the zig-zag line. This process is repeated. Slight modification is needed when we approach the stage where we want to express the quantities in the column headed by $A^{\mu+1}$.

Lemma B.IV. The conditions (B3) and (B4) in Lemma B.III are, in fact, satisfied by the functions

$$A=I(I+1)-\tfrac{1}{4}S^2$$
$$B=S.$$

Proof. (i) From Lemma B.II we see that all the points of Fig. 2 satisfy the relation

$$\prod_{i=1}^{\nu+1}(I(I+1)-\tfrac{1}{4}S^2+b_iS+c_i)=0.$$

This is condition (B3). (ii) To prove Eq. (B4) we divide the set of points in Fig. 2 into two sets S_i, S_i' $(i=0, 1, \cdots, \nu)$ as follows:

Let S take the distinct values $s_1, s_2, \cdots s_{\mu+\nu+1}$ expressed as a monotonically increasing sequence. For $i\neq 0$, suppose the set S_i consists of all the points having S as any of

$$s_1, s_2, \cdots s_i, \quad s_{\mu+\nu-i+2}, \cdots s_{\mu+\nu}, s_{\mu+\nu+1},$$

and let S_0 be the null set. The set S_i' consists of the remaining points in the figure. Consider first $i\neq 0$. It is clear from Lemma B.I that the set S_i consists of $i(i+1)$ points. S_i can, therefore, determine a set of values of the $i(i+1)$ ratios of the $(i(i+1)+1)$ constants a_{rs} such that

$$d(I(I+1)-\tfrac{1}{4}S^2)^i + \sum_{\substack{2r+s<2i \\ r,s>0}} a_{rs}[I(I+1)-\tfrac{1}{4}S^2]^rS^s=0 \quad \text{(B5)}$$

is satisfied by all the points of the set. Here d is necessarily nonzero, for if it were zero, Eq. (B5) which is now of at most $i-1$ degree in $I(I+1)$, cannot satisfy all the i distinct points with $S=s_i$ because the corresponding $I(I+1)$ are necessarily distinct and positive definite.

Thus, all the points of the figure satisfy the set of equations

$$\left\{[I(I+1)-\tfrac{1}{4}S^2]^i+\sum_{\substack{2r+s<2i \\ r,s>0}}\frac{a_{rs}}{d}[I(I+1)-\tfrac{1}{4}S^2]^rS^s\right\} \times(S-s_{i+1})\cdots,(S-s_{\mu+\nu-i+1})=0. \quad \text{(B6)}$$

When $i=0$ we have instead

$$(S-s_1)(S-s_2)\cdots,(S-s_{\mu+\nu+1})=0. \quad \text{(B7)}$$

These give condition (B4).